反射テスト　積分　定積分　部分積分法　02

1.　次の計算をせよ.（ S 級 4 分, A 級 6 分, B 級 9 分, C 級 12 分 ）

(1)　$\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x\sin x\cos x\,dx$

(2)　$\displaystyle\int_{0}^{\pi} e^{x}\cos x\,dx$

2. 次の計算をせよ.（ S 級 4 分, A 級 6 分, B 級 9 分, C 級 12 分 ）

(1) $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} x \sin\frac{x}{2}\cos\frac{x}{2}\,dx$

(2) $\displaystyle\int_{0}^{\pi} e^{x}\sin x\,dx$

反射テスト　積分　定積分　部分積分法　02　解答解説

1. 次の計算をせよ.（ S 級 4 分, A 級 6 分, B 級 9 分, C 級 12 分 ）

★ 部分積分法 $\displaystyle \int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$

(1) $\displaystyle \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x\sin x\cos x\,dx$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x\cdot\frac{1}{2}\sin 2x\,dx \quad \leftarrow ☆1$$
$$= \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{4}} x\sin 2x\,dx \quad \leftarrow ☆2$$
$$= \int_0^{\frac{\pi}{4}} x\sin 2x\,dx$$
$$= \int_0^{\frac{\pi}{4}} x\left(-\frac{1}{2}\cos 2x\right)'dx$$
$$= \left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (x)'\cdot\left(-\frac{1}{2}\cos 2x\right)dx$$
$$= -\frac{1}{2}\left[x\cos 2x\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}}\cos 2x\,dx$$
$$= -\frac{1}{2}\left[x\cos 2x\right]_0^{\frac{\pi}{4}} + \frac{1}{4}\left[\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= -\frac{1}{2}\cdot\frac{\pi}{4}\cos\left(2\cdot\frac{\pi}{4}\right) + \frac{1}{4}\cdot\sin\left(2\cdot\frac{\pi}{4}\right)$$
$$= \boldsymbol{\frac{1}{4}}$$

☆1　$2\sin x\cos x = \sin 2x$

☆2　奇関数・奇関数 = 偶関数

(2) $\displaystyle \int_0^{\pi} e^x\cos x\,dx$

$$\begin{cases} I = \displaystyle\int e^x\sin x\,dx \\ J = \displaystyle\int e^x\cos x\,dx \end{cases}$$ とおく.

$$I = \int e^x\cdot(-\cos x)'\,dx$$
$$= e^x\cdot(-\cos x) - \int (e^x)'\cdot(-\cos x)\,dx$$
$$= -e^x\cos x + \int e^x\cos x\,dx$$
$$= -e^x\cos x + J$$

$$J = \int e^x\cdot(\sin x)'\,dx$$
$$= e^x\cdot\sin x - \int (e^x)'\cdot\sin x\,dx$$
$$= e^x\sin x - \int e^x\sin x\,dx$$
$$= e^x\sin x - I$$

$$\therefore \begin{cases} I = -e^x\cos x + J \\ J = e^x\sin x - I \end{cases}$$

これを I, J について解くと,

$$\begin{cases} I = \dfrac{1}{2}e^x(\sin x - \cos x) \\ J = \dfrac{1}{2}e^x(\sin x + \cos x) \end{cases}$$

よって,

$$与式 = \frac{1}{2}\left[e^x(\sin x + \cos x)\right]_0^{\pi}$$
$$= \frac{1}{2}\left\{e^{\pi}(\sin\pi + \cos\pi) - e^0(\sin 0 + \cos 0)\right\}$$
$$= \boldsymbol{-\frac{1}{2}e^{\pi} - \frac{1}{2}}$$

2. 次の計算をせよ.（ S 級 4 分, A 級 6 分, B 級 9 分, C 級 12 分 ）

(1) $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} x \sin\frac{x}{2}\cos\frac{x}{2}\,dx$

$$
\begin{aligned}
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} x\cdot\frac{1}{2}\sin x\,dx \quad \leftarrow☆1\\
&= \tfrac{1}{2}\cdot 2\int_{0}^{\frac{\pi}{6}} x\sin x\,dx \quad \leftarrow☆2\\
&= \int_{0}^{\frac{\pi}{6}} x\sin x\,dx\\
&= \int_{0}^{\frac{\pi}{6}} x\,(-\cos x)'\,dx\\
&= \left[x\,(-\cos x)\right]_0^{\frac{\pi}{6}} - \int_{0}^{\frac{\pi}{6}} (x)'\cdot(-\cos x)\,dx\\
&= -\left[x\cos x\right]_0^{\frac{\pi}{6}} + \int_{0}^{\frac{\pi}{6}} \cos x\,dx\\
&= \left[-x\cos x + \sin x\right]_0^{\frac{\pi}{6}}\\
&= \left(-\frac{\pi}{6}\cos\frac{\pi}{6} + \sin\frac{\pi}{6}\right) - (-0\cos 0 + \sin 0)\\
&= -\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\\
&= \boldsymbol{\frac{1}{2} - \frac{\sqrt{3}\pi}{12}}
\end{aligned}
$$

☆1　$2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x$

☆2　奇関数・奇関数 = 偶関数

(2) $\displaystyle\int_{0}^{\pi} e^x \sin x\,dx$

$$
\begin{cases}
I = \displaystyle\int e^x\sin x\,dx\\
J = \displaystyle\int e^x\cos x\,dx
\end{cases}
$$
とおく.

$$
\begin{aligned}
I &= \int e^x\cdot(-\cos x)'\,dx\\
&= e^x\cdot(-\cos x) - \int (e^x)'\cdot(-\cos x)\,dx\\
&= -e^x\cos x + \int e^x\cos x\,dx\\
&= -e^x\cos x + J
\end{aligned}
$$

$$
\begin{aligned}
J &= \int e^x\cdot(\sin x)'\,dx\\
&= e^x\cdot\sin x - \int (e^x)'\cdot\sin x\,dx\\
&= e^x\sin x - \int e^x\sin x\,dx\\
&= e^x\sin x - I
\end{aligned}
$$

$$
\therefore\quad
\begin{cases}
I = -e^x\cos x + J\\
J = e^x\sin x - I
\end{cases}
$$

これを I, J について解くと,

$$
\begin{cases}
I = \dfrac{1}{2}e^x(\sin x - \cos x)\\
J = \dfrac{1}{2}e^x(\sin x + \cos x)
\end{cases}
$$

よって,

$$
\begin{aligned}
\text{与式} &= \frac{1}{2}\left[e^x(\sin x - \cos x)\right]_0^{\pi}\\
&= \frac{1}{2}\left\{e^{\pi}(\sin\pi - \cos\pi) - e^0(\sin 0 - \cos 0)\right\}\\
&= \frac{1}{2}\left[e^{\pi}\{0-(-1)\} - 1(0-1)\right]\\
&= \boldsymbol{\frac{1}{2}e^{\pi} + \frac{1}{2}}
\end{aligned}
$$