反射テスト　三角関数　合成　02

1.　次の式を合成して，sin の単項式で表せ.（ S 級 3 分 30 秒，A 級 5 分，B 級 7 分，C 級 9 分 ）

(1)　$\sin x + \cos x$

(2)　$\sin x + \sqrt{3}\cos x$

(3)　$\sqrt{6}\sin x - \sqrt{2}\cos x$

(4)　$a\cos x - \sin x$　（ $a > 0$ ）

(5)　$\sin x - \cos\left(x + \frac{\pi}{6}\right)$

(6)　$\sqrt{2}\sin\left(x + \frac{13}{12}\pi\right) + \sqrt{2}\cos\left(x + \frac{\pi}{12}\right)$

2. 次の式を合成して，sin の単項式で表せ.（ S 級 4 分 30 秒，A 級 6 分 20 秒，B 級 9 分，C 級 12 分 ）

(1) $6\sin x + 6\cos x$

(2) $\dfrac{1}{\sqrt{3}}\sin x - \cos x$

(3) $-\sqrt{10}\sin x + \sqrt{10}\cos x$

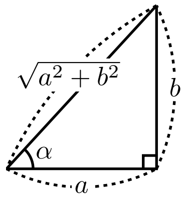
(4) $(b-a)\sin x + (a+b)\cos x$ （ $0 < a < b$ ）

(5) $2\sin\left(x - \dfrac{5}{6}\pi\right) + 2\cos x$

(6) $\sin\left(x - \dfrac{5}{12}\pi\right) + \sqrt{3}\cos\left(x - \dfrac{\pi}{12}\right)$

反射テスト　三角関数　合成　02　解答解説

1.　次の式を合成して，$\sin$ の単項式で表せ．(S 級 3 分 30 秒，A 級 5 分，B 級 7 分，C 級 9 分)



★ 三角関数の合成　（加法定理の逆）

$\boldsymbol{a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\alpha)}$　ただし $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

証明概略　左図のような三角形を考え，加法定理 $\cos\alpha\sin x + \sin\alpha\cos x = \sin(x+\alpha)$ を用いる

☆合成後，$x = 0°$ を代入して確かめる癖をつけるとよい．

(1)　$\sin x + \cos x$

$\sqrt{1^2+1^2} = \sqrt{2}$

$\begin{cases} \cos\alpha = \frac{1}{\sqrt{2}} \\ \sin\alpha = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \quad \alpha = \frac{\pi}{4}$

与式 $= \boldsymbol{\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)}$

(2)　$\sin x + \sqrt{3}\cos x$

$\sqrt{1^2 + \sqrt{3}^2} = 2$

$\begin{cases} \cos\alpha = \frac{1}{2} \\ \sin\alpha = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow \quad \alpha = \frac{\pi}{3}$

与式 $= \boldsymbol{2\sin\left(x + \frac{\pi}{3}\right)}$

(3)　$\sqrt{6}\sin x - \sqrt{2}\cos x$

$\sqrt{\sqrt{6}^2 + (-\sqrt{2})^2} = 2\sqrt{2}$

$\begin{cases} \cos\alpha = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2} \\ \sin\alpha = \frac{-\sqrt{2}}{2\sqrt{2}} = -\frac{1}{2} \end{cases} \Rightarrow \quad \alpha = -\frac{\pi}{6}$

与式 $= \boldsymbol{2\sqrt{2}\sin\left(x - \frac{\pi}{6}\right)}$

☆別解　$2\sqrt{2}\sin\left(x + \frac{11}{6}\pi\right)$

(4)　$a\cos x - \sin x$　($a > 0$)

$= -\sin x + a\cos x$

$\sqrt{(-1)^2 + a^2} = \sqrt{a^2+1}$

与式 $= \boldsymbol{\sqrt{a^2+1}\sin(x+\alpha)}$

ただし $\boldsymbol{\alpha}$ は $\boldsymbol{\begin{cases} \cos\alpha = -\frac{1}{\sqrt{a^2+1}} \\ \sin\alpha = \frac{a}{\sqrt{a^2+1}} \end{cases}}$ **を満たす．**

(5)　$\sin x - \cos\left(x + \frac{\pi}{6}\right)$

$$
\begin{aligned}
&= \sin x - \left(\cos x\cos\tfrac{\pi}{6} - \sin x\sin\tfrac{\pi}{6}\right) \\
&= \sin x - \left(\tfrac{\sqrt{3}}{2}\cos x - \tfrac{1}{2}\sin x\right) \\
&= \tfrac{3}{2}\sin x - \tfrac{\sqrt{3}}{2}\cos x
\end{aligned}
$$

$\sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}$

$\begin{cases} \cos\alpha = \frac{\frac{3}{2}}{\sqrt{3}} = \frac{\sqrt{3}}{2} \\ \sin\alpha = -\frac{\frac{\sqrt{3}}{2}}{\sqrt{3}} = -\frac{1}{2} \end{cases} \Rightarrow \quad \alpha = -\frac{\pi}{6}$

与式 $= \boldsymbol{\sqrt{3}\sin\left(x - \frac{\pi}{6}\right)}$

☆別解　$\sqrt{3}\sin\left(x + \frac{11}{6}\pi\right)$

(6)　$\sqrt{2}\sin\left(x + \frac{13}{12}\pi\right) + \sqrt{2}\cos\left(x + \frac{\pi}{12}\right)$

$$
\begin{aligned}
&= \sqrt{2}\sin\left\{\left(x + \tfrac{\pi}{12}\right) + \pi\right\} + \sqrt{2}\cos\left(x + \tfrac{\pi}{12}\right) \\
&= \sqrt{2}\left\{\sin\left(x + \tfrac{\pi}{12}\right)\cos\pi + \cos\left(x + \tfrac{\pi}{12}\right)\sin\pi\right\} \\
&\qquad + \sqrt{2}\cos\left(x + \tfrac{\pi}{12}\right) \\
&= -\sqrt{2}\sin\left(x + \tfrac{\pi}{12}\right) + \sqrt{2}\cos\left(x + \tfrac{\pi}{12}\right) \\
&= \sqrt{2}\left\{-\sin\left(x + \tfrac{\pi}{12}\right) + \cos\left(x + \tfrac{\pi}{12}\right)\right\} \\
&= \sqrt{2}\cdot\sqrt{2}\sin\left\{\left(x + \tfrac{\pi}{12}\right) + \tfrac{3}{4}\pi\right\} \quad \leftarrow☆\ 1(4) \\
&= \boldsymbol{2\sin\left(x + \tfrac{5}{6}\pi\right)}
\end{aligned}
$$

☆　$\sin(\theta + \pi) = -\sin\theta$

を使うと早い．他にも方法はある．

☆別解　$\begin{cases} -2\sin\left(x + \frac{11}{6}\pi\right) \\ -2\sin\left(x - \frac{\pi}{6}\right) \\ 2\sin\left(x - \frac{7}{6}\pi\right) \end{cases}$

2. 次の式を合成して，sin の単項式で表せ.（ S 級 4 分 30 秒，A 級 6 分 20 秒，B 級 9 分，C 級 12 分 ）

(1) $6\sin x + 6\cos x$

$= 6(\sin x + \cos x)$

$\sqrt{1^2+1^2} = \sqrt{2}$

$\begin{cases} \cos\alpha = \frac{1}{\sqrt{2}} \\ \sin\alpha = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \quad \alpha = \frac{\pi}{4}$

与式 $= \boldsymbol{6\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}$

(2) $\dfrac{1}{\sqrt{3}}\sin x - \cos x$

$= \frac{1}{\sqrt{3}}(\sin x - \sqrt{3}\cos x)$

$\sqrt{1^2+(-\sqrt{3})^2} = 2$

$\begin{cases} \cos\alpha = \frac{1}{2} \\ \sin\alpha = -\frac{\sqrt{3}}{2} \end{cases} \Rightarrow \quad \alpha = -\frac{\pi}{3}$

与式 $= \dfrac{2}{\sqrt{3}}\sin\left(x-\frac{\pi}{3}\right)$

$= \boldsymbol{\dfrac{2\sqrt{3}}{3}\sin\left(x-\frac{\pi}{3}\right)}$

(3) $-\sqrt{10}\sin x + \sqrt{10}\cos x$

$= -\sqrt{10}(\sin x - \cos x)$

$\sqrt{1^2+(-1)^2} = \sqrt{2}$

$\begin{cases} \cos\alpha = \frac{1}{\sqrt{2}} \\ \sin\alpha = -\frac{1}{\sqrt{2}} \end{cases} \Rightarrow \quad \alpha = -\frac{\pi}{4}$

与式 $= -\sqrt{10}\cdot\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

$= \boldsymbol{-2\sqrt{5}\sin\left(x-\frac{\pi}{4}\right)}$

$= \boldsymbol{2\sqrt{5}\sin\left(x+\frac{3}{4}\pi\right)}$

☆別解 $-2\sqrt{5}\sin\left(x+\frac{7}{4}\pi\right)$

(4) $(b-a)\sin x + (a+b)\cos x$ （ $0 < a < b$ ）

$\sqrt{(a-b)^2+(a+b)^2} = \sqrt{2\,(a^2+b^2)}$

与式 $= \boldsymbol{\sqrt{2\,(a^2+b^2)}\sin(x+\alpha)}$

ただし $\boldsymbol{\alpha}$ は $\boldsymbol{\begin{cases} \cos\alpha = \frac{b-a}{\sqrt{2(a^2+b^2)}} \\ \sin\alpha = \frac{a+b}{\sqrt{2(a^2+b^2)}} \end{cases}}$ **を満たす.**

(5) $2\sin\left(x-\dfrac{5}{6}\pi\right) + 2\cos x$

$= 2\left(\sin x\cos\frac{5}{6}\pi - \cos x\sin\frac{5}{6}\pi\right) + 2\cos x$

$= 2\left(-\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right) + 2\cos x$

$= -\sqrt{3}\sin x + \cos x$

$\sqrt{(-\sqrt{3})^2+1^2} = 2$

$\begin{cases} \cos\alpha = -\frac{\sqrt{3}}{2} \\ \sin\alpha = 12 \end{cases} \Rightarrow \quad \alpha = \frac{5}{6}\pi$

与式 $= \boldsymbol{2\sin\left(x+\frac{5}{6}\pi\right)}$

(6) $\sin\left(x-\dfrac{5}{12}\pi\right) + \sqrt{3}\cos\left(x-\dfrac{\pi}{12}\right)$

$= \sin\left(x-\frac{5}{12}\pi\right) + \sqrt{3}\cos\left\{\left(x-\frac{5}{12}\pi\right)+\frac{\pi}{3}\right\}$

$= \sin\left(x-\frac{5}{12}\pi\right)$

$\quad + \sqrt{3}\left\{\cos\left(x-\frac{5}{12}\pi\right)\cos\frac{\pi}{3} - \sin\left(x-\frac{5}{12}\pi\right)\sin\frac{\pi}{3}\right\}$

$= \sin\left(x-\frac{5}{12}\pi\right) + \frac{\sqrt{3}}{2}\cos\left(x-\frac{5}{12}\pi\right) - \frac{3}{2}\sin\left(x-\frac{5}{12}\pi\right)$

$= -\frac{1}{2}\sin\left(x-\frac{5}{12}\pi\right) + \frac{\sqrt{3}}{2}\cos\left(x-\frac{5}{12}\pi\right)$

$\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = 1$

$\begin{cases} \cos\alpha = -\frac{1}{2} \\ \sin\alpha = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow \quad \alpha = \frac{2}{3}\pi$

与式 $= 1\sin\left\{\left(x-\frac{5}{12}\pi\right)+\frac{2}{3}\pi\right\}$

$= \boldsymbol{\sin\left(x+\frac{\pi}{4}\right)}$

☆他にも方法はいろいろある.

☆別解 $\begin{cases} -\sin\left(x+\frac{5}{4}\pi\right) \\ -\sin\left(x-\frac{3}{4}\pi\right) \\ \sin\left(x-\frac{7}{4}\pi\right) \end{cases}$